

# Nonperturbative RG analysis of five-dimensional $O(N)$ models with cubic interactions

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We reconsider critical properties of  $O(N)$  scalar models with cubic interactions in  $d > 4$  dimensions using functional renormalization group equations. Working at next-to-leading order in the derivative expansion, we find non-trivial IR fixed points at small and intermediate  $N$  from beta functions for relevant cubic terms. The putative fixed point at large  $N$  suggested recently by higher spin holography and the  $\varepsilon$ -expansion is also discussed, with an emphasis on stability of the effective potential.

## I. INTRODUCTION

Theoretical understanding of critical phenomena and universality in the framework of renormalization group is a milestone in modern theoretical physics [1–3]. Among others the  $O(N)$  models have been thoroughly investigated with various methods [e.g., the celebrated  $\varepsilon$ -expansion,  $1/N$  expansion, high-temperature expansion, Monte Carlo simulation and the functional renormalization group (FRG)] and nowadays there seems to be a solid theoretical ground for our understanding of  $O(N)$ -symmetric critical points in  $2 < d < 4$  dimensions. In the upper critical dimension  $d = 4$  the Wilson-Fisher fixed point merges with the Gaussian fixed point and so far no nontrivial stable fixed point with physically acceptable properties has been found in  $d \geq 4$ , in agreement with the Ginzburg criterion for a mean-field theory. In addition, there are even rigorous proofs of triviality for  $N = 1$  and 2 [4, 5]. Recent works by Fei et al. [6, 7] suggested, however, possible existence of a unitary non-Gaussian fixed point in  $4 < d < 6$  at least for sufficiently large  $N$ . Their argument is supported by higher-spin AdS/CFT dualities [8, 9]. While the original approach [6, 7] (see also [10, 11]) was based on the  $\varepsilon$ -expansion from  $d = 6$  dimensions, it would be quite desirable to perform independent checks with nonperturbative methods. In this regard, the conformal bootstrap approach [12–14] and FRG [15–17] have so far yielded contrasting results as to the existence of a new fixed point: the former supports the claim of [6, 7] whereas the latter does not. In this paper, we investigate the putative critical point in  $d = 5$  by means of FRG, not based on the conventional  $(\phi_i \phi_i)^2$ -type formulation with  $N$  scalars but on the cubic  $O(N)$  model with  $N + 1$  scalars [6]. One of the advantages of FRG is that one can directly work in  $d = 5$  with no need for dimensional continuation from  $d = 4$  or 6. Our analysis suggests that no nontrivial critical point exists at large  $N$ , in accordance with [15–17].

This paper is structured as follows. In Sec. II we define the model, explain the effective average action approach and present the flow equation at next-to-leading order in the derivative expansion. The structure of the flow at finite  $N$  is sketched. In Sec. III the flow in the large- $N$  limit is discussed and compared with the flow from  $\varepsilon$ -expansion. We conclude in Sec. IV. In Appendix the

derivation of the flow equations is outlined.

## II. RG EQUATION FOR A CUBIC $O(N)$ THEORY

The functional renormalization group (FRG) is a powerful nonperturbative method to solve problems with multiple scales in quantum field theory and statistical physics. In this approach, we consider the scale-dependent effective average action,  $\Gamma_k[\phi]$ , which obeys the exact flow equation [18–21]

$$\partial_k \Gamma_k = \frac{1}{2} \text{STr} \left\{ (\Gamma_k^{(2)} + R_k)^{-1} \partial_k R_k \right\}. \quad (1)$$

Here  $\text{STr}$  and  $R_k$  denotes a functional trace in superspace and a regulator of the flow, respectively.  $\Gamma_{k=\Lambda}$  equals the classical action on the microscopic scale, while  $\Gamma_{k=0}$  is nothing but the full 1PI effective action. For more details, we refer the reader to [22–29].

In this work we apply the FRG method to analyze the cubic  $O(N)$  model [6]

$$S = \int d^d x \left[ \frac{1}{2} (\partial_\mu \phi_i)^2 + \frac{1}{2} (\partial_\mu \sigma)^2 + \frac{g_1}{2} \sigma \phi_i \phi_i + \frac{g_2}{6} \sigma^3 \right], \quad (2)$$

which is perturbatively renormalizable in  $d = 6$ . The index  $i$  runs from 1 to  $N$  and we leave  $d$  and  $N$  arbitrary at this stage. Despite the precarious cubic potential, scalar theories like (2) have long been investigated due to their relevance to the Yang-Lee edge singularity [30, 31], percolation problems [31, 32] and more recently, a six-dimensional generalization of the  $a$ -theorem [33, 34] and  $\mathcal{PT}$ -symmetric field theories [35, 36].

Let us recall that, in the conventional setup of the  $O(N)$  vector model, the coupling for  $(\phi_i \phi_i)^2$  is marginal in  $d = 4$  and irrelevant in  $d > 4$ . Then a nontrivial fixed point, if any, should appear as a UV fixed point [16, 17]. By contrast, the action (2) has two cubic couplings that are marginal in  $d = 6$ , so we may look for a nontrivial IR fixed point in  $d < 6$ . This situation is reminiscent of the Wilson-Fisher fixed point in  $d = 3$  which can be identified either from a nonlinear sigma model in  $d = 2 + \varepsilon$  as a UV fixed point, or from a quartic scalar theory in  $d = 4 - \varepsilon$  as an IR fixed point [2].

While the flow equation (1) itself is exact, one needs to project it to a finite-dimensional functional space to make practical calculations feasible. In this work, we employ the Ansatz

$$\Gamma_k[\phi, \sigma] = \int d^d x \left[ \frac{Y_k}{2} (\partial_\mu \phi_i)^2 + \frac{Z_k}{2} (\partial_\mu \sigma)^2 + U_k(\rho, \sigma) \right] \quad (3)$$

with  $\rho \equiv \frac{1}{2} \phi_i \phi_i$  for the truncated effective action at the scale  $k$ . We could also add a term  $(\partial_\mu \rho)^2$  that contributes to the difference of anomalous dimensions for the radial mode and the Nambu-Goldstone modes, but here it is omitted due to its high canonical dimension. The factors  $Y_k$  and  $Z_k$  are the wave function renormalization for  $\phi_i$  and  $\sigma$ , respectively, and  $U_k$  is the running effective potential. The approximation for (3), which is at the next-to-leading order in the derivative expansion, is called the improved local potential approximation (LPA'); when  $Y_k = Z_k \equiv 1$  and only  $U_k$  is running, this is the leading order in the derivative expansion and is called LPA. Both

LPA and LPA' have been successful in describing various critical phenomena [22–25, 29, 37]. Although there is no small parameter that controls the expansion of the truncated effective action, it is known that LPA and LPA' work better when the anomalous dimension of fields is numerically small. In the large- $N$  limit of the quartic  $O(N)$  vector model where the anomalous dimension vanishes, LPA becomes *exact* for the effective potential [2, 38, 39].

In this work we employ the optimized regulator devised by Litim [40, 41]

$$\left\{ \begin{matrix} R_k^\phi(p) \\ R_k^\sigma(p) \end{matrix} \right\} = \left\{ \begin{matrix} Y_k \\ Z_k \end{matrix} \right\} \times (k^2 - p^2) \Theta(k^2 - p^2) \quad \text{for} \quad \left\{ \begin{matrix} \phi_i \\ \sigma \end{matrix} \right\}, \quad (4)$$

where  $\Theta$  is the Heaviside step function. The flow equations for  $Y_k$ ,  $Z_k$  and  $U_k$  can now be obtained straightforwardly by plugging (3) and (4) into (1). Full details of the derivation are presented in Appendix. Introducing the logarithmic scale  $t \equiv \log(k/\Lambda)$ , we obtain

$$\partial_t U_k = \mu_d k^{d+2} \left[ \frac{Y_k k^2 + \frac{\partial U_k}{\partial \rho} + 2\rho \frac{\partial^2 U_k}{\partial \rho^2}}{(Z_k k^2 + \frac{\partial^2 U_k}{\partial \sigma^2})(Y_k k^2 + \frac{\partial U_k}{\partial \rho} + 2\rho \frac{\partial^2 U_k}{\partial \rho^2}) - 2\rho (\frac{\partial^2 U_k}{\partial \rho \partial \sigma})^2} \left( 1 - \frac{\eta_\sigma}{d+2} \right) Z_k \right. \\ \left. + \left\{ \frac{Z_k k^2 + \frac{\partial^2 U_k}{\partial \sigma^2}}{(Z_k k^2 + \frac{\partial^2 U_k}{\partial \sigma^2})(Y_k k^2 + \frac{\partial U_k}{\partial \rho} + 2\rho \frac{\partial^2 U_k}{\partial \rho^2}) - 2\rho (\frac{\partial^2 U_k}{\partial \rho \partial \sigma})^2} + \frac{N-1}{Y_k k^2 + \frac{\partial U_k}{\partial \rho}} \right\} \left( 1 - \frac{\eta_\phi}{d+2} \right) Y_k \right], \quad (5)$$

$$\eta_\phi \equiv -\partial_t \log Y_k \quad (6)$$

$$= 2\mu_d k^{d+2} \left\langle \frac{\partial^2 U_k}{\partial \sigma \partial \rho} \right\rangle^2 \frac{Z_k}{\left( Y_k k^2 + \left\langle \frac{\partial U_k}{\partial \rho} \right\rangle \right)^2 \left( Z_k k^2 + \left\langle \frac{\partial^2 U_k}{\partial \sigma^2} \right\rangle \right)}, \quad (7)$$

$$\eta_\sigma \equiv -\partial_t \log Z_k \quad (8)$$

$$= \mu_d k^{d+2} \left\langle \frac{\partial^3 U_k}{\partial \sigma^3} \right\rangle^2 \frac{Z_k}{\left( Z_k k^2 + \left\langle \frac{\partial^2 U_k}{\partial \sigma^2} \right\rangle \right)^4} + N\mu_d k^{d+2} \left\langle \frac{\partial^2 U_k}{\partial \sigma \partial \rho} \right\rangle^2 \frac{Y_k^2/Z_k}{\left( Y_k k^2 + \left\langle \frac{\partial U_k}{\partial \rho} \right\rangle \right)^4}, \quad (9)$$

where

$$\mu_d \equiv \frac{1}{(4\pi)^{d/2} \Gamma(\frac{d}{2} + 1)} \quad (10)$$

and the bracket  $\langle \dots \rangle$  denotes the value in a fixed background  $(\sigma(x), \phi_i(x)) = (\sigma_0, \vec{0})$ . Although the anomalous dimensions in LPA' are sometimes evaluated at the running minimum of the effective potential for better convergence [37, 42], here we set  $\phi_i = \vec{0}$  for a technical reason

and leave  $\sigma_0$  arbitrary at this stage.

To investigate the scaling behavior near the fixed point it is convenient to make all variables dimensionless by proper powers of  $k$ . We thus define

$$u_t(r, s) = k^{-d} U_k(\rho, \sigma), \quad (11a)$$

$$r = k^{2-d} Y_k \rho, \quad (11b)$$

$$s = k^{\frac{2-d}{2}} \sqrt{Z_k} \sigma, \quad (11c)$$

which leads to the dimensionless flow equations

$$\partial_t u_t + du_t + (2-d-\eta_\phi) r \partial_r u_t + \frac{1}{2} (2-d-\eta_\sigma) s \partial_s u_t$$

$$= \mu_d \left[ \frac{1 + \partial_r u_t + 2r \partial_r^2 u_t}{(1 + \partial_s^2 u_t)(1 + \partial_r u_t + 2r \partial_r^2 u_t) - 2r(\partial_r \partial_s u_t)^2} \left(1 - \frac{\eta_\sigma}{d+2}\right) + \left\{ \frac{1 + \partial_s^2 u_t}{(1 + \partial_s^2 u_t)(1 + \partial_r u_t + 2r \partial_r^2 u_t) - 2r(\partial_r \partial_s u_t)^2} + \frac{N-1}{1 + \partial_r u_t} \right\} \left(1 - \frac{\eta_\phi}{d+2}\right) \right], \quad (12)$$

$$\eta_\phi = 2\mu_d \frac{\langle \partial_r \partial_s u_t \rangle^2}{(1 + \langle \partial_r u_t \rangle)^2 (1 + \langle \partial_s^2 u_t \rangle)^2}, \quad (13)$$

$$\eta_\sigma = \mu_d \left[ \frac{\langle \partial_s^3 u_t \rangle^2}{(1 + \langle \partial_s^2 u_t \rangle)^4} + N \frac{\langle \partial_r \partial_s u_t \rangle^2}{(1 + \langle \partial_r u_t \rangle)^4} \right]. \quad (14)$$

As a small check, notice that if we neglect the  $s$ -dependence of  $u_t$ , we find  $\eta_\phi = \eta_\sigma = 0$  and recover the flow equation for  $u_t$  in the quartic  $O(N)$  model with no  $\sigma$  field [41].<sup>1</sup>

To make a comparison with the  $\varepsilon$ -expansion around  $d = 6$ , let us substitute a simplistic Ansatz [cf. (2)]

$$u_t(r, s) = \hat{g}_1(t)rs + \frac{\hat{g}_2(t)}{6}s^3 \quad (15)$$

into (12) and expand the RHS in powers of  $r$  and  $s$ . Furthermore we choose to evaluate  $\langle \dots \rangle$  in (13) and (14) at  $\sigma_0 = 0$ . This yields the beta functions of  $\hat{g}_1$  and  $\hat{g}_2$ ,

$$\frac{d\hat{g}_1}{dt} = -\frac{6-d-2\eta_\phi-\eta_\sigma}{2}\hat{g}_1 - 2\mu_d\hat{g}_1^2 \left[ \left(3 - \frac{2\eta_\phi+\eta_\sigma}{d+2}\right)\hat{g}_1 + \left(3 - \frac{\eta_\phi+2\eta_\sigma}{d+2}\right)\hat{g}_2 \right], \quad (16a)$$

$$\frac{d\hat{g}_2}{dt} = -\frac{6-d-3\eta_\sigma}{2}\hat{g}_2 - 6\mu_d \left[ \left(1 - \frac{\eta_\phi}{d+2}\right)N\hat{g}_1^3 + \left(1 - \frac{\eta_\sigma}{d+2}\right)\hat{g}_2^3 \right], \quad (16b)$$

and

$$\eta_\phi = 2\mu_d\hat{g}_1^2, \quad \eta_\sigma = \mu_d(N\hat{g}_1^2 + \hat{g}_2^2). \quad (17)$$

If we suppress  $\eta_\phi$  and  $\eta_\sigma$  in the square brackets of (16), insert  $d = 6 - \varepsilon$  in the first term of (16) and set  $\mu_d = \mu_6 = \frac{1}{(4\pi)^{d/2}\Gamma(\frac{d}{2}+1)}\Big|_{d=6} = \frac{1}{6(4\pi)^3}$ , then  $d\hat{g}_1/dt$  and  $d\hat{g}_2/dt$  exactly match the beta functions from the  $\varepsilon$ -expansion at one loop [6]. Also  $\eta_\phi/2$  and  $\eta_\sigma/2$  in (17) exactly match the anomalous dimensions of  $\phi$  and  $\sigma$  in [6].<sup>2</sup> However, one difference from the  $\varepsilon$ -expansion is that  $\eta_\phi$  and  $\eta_\sigma$  multiply  $\hat{g}_1^3$ ,  $\hat{g}_1^2\hat{g}_2$  and  $\hat{g}_2^3$  in (16). This reflects

the fact that higher-order contributions are incorporated differently in the loop expansion and FRG. In Fig. 1 we display the flow diagram of (16) for  $d = 5$  and  $N = 2$ . Intriguingly, besides the Gaussian fixed point at the origin  $(0, 0)$ , there are two nontrivial IR-stable fixed points  $A$  and  $B$  that are absent in the  $\varepsilon$ -expansion at one loop. The anomalous dimensions at these fixed points are

$$\begin{aligned} A : (\eta_\phi, \eta_\sigma) &= (3.67, \quad 6.58), \\ B : (\eta_\phi, \eta_\sigma) &= (0, \quad 5.36), \end{aligned} \quad (18)$$

respectively. The point  $B$  is present for any  $N \geq 2$ , whereas the point  $A$  disappears for  $N \geq 19$ . The large- $N$  fixed point taken up in [6, 7] comes into existence only for  $N \gtrsim 820$ , which is slightly below the threshold  $\sim 1039$  in the one-loop  $\varepsilon$ -expansion [6]. The question one must ask is whether  $A$  and  $B$  do represent physical critical points or not. In this regard we have to recognize that the anomalous dimensions (18) at  $A$  and  $B$  are dangerously large and threaten the validity of LPA'. Note also that the presence of *multiple* IR-stable fixed points leads to a bewildering consequence that two systems in the same dimension and sharing the same symmetry may exhibit different universal behaviors without fine-tuning, depending on which basin of attraction the initial parameters fall in. This exotic situation is not expected to

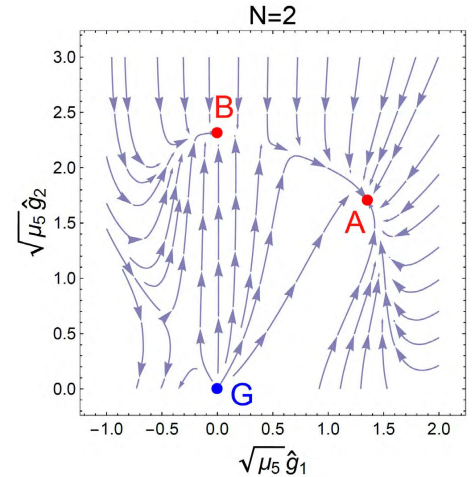


FIG. 1. RG flow towards IR with  $N = 2$  and  $d = 5$  for the minimally truncated Ansatz (15). The flow diagram is symmetric under  $(\hat{g}_1, \hat{g}_2) \leftrightarrow (-\hat{g}_1, -\hat{g}_2)$  and only the region with  $\hat{g}_2 \geq 0$  is shown. The blobs  $A$  and  $B$  represent IR-stable fixed points, while the blob  $G$  is the Gaussian fixed point which is unstable in IR. There are two more unstable fixed points in the figure (not shown).

<sup>1</sup> There remains an irrelevant constant in the RHS of (12), which represents the contribution of the free massless scalar  $\sigma$ .

<sup>2</sup> Coincidence of the perturbatively expanded FRG and the one-loop  $\varepsilon$ -expansion has been reported for a  $U(2) \times U(2)$  scalar model [43].

arise in physically sound systems. Based on these observations, we would like to take a conservative point of view that the fixed points  $A$  and  $B$  are artifacts of the truncation (15).<sup>3</sup> One way to test this idea would be by extending the truncation of  $u_t$  to higher orders and check stability of  $A$  and  $B$ , taking carefully into account a number of subtleties associated with the polynomial truncation method [17, 42, 44, 45].

### III. LARGE $N$

Next we turn to the analysis at  $N \gg 1$ . In this limit the flow equations are simplified. For the counting

$$u_t(r, s) \sim r \sim N \quad \text{and} \quad s \sim \sqrt{N}, \quad (19)$$

one obtains, at leading order,

$$\begin{aligned} \partial_t u_t + du_t + (2-d)r\partial_r u_t + \frac{1}{2}(2-d-\eta_\sigma)s\partial_s u_t \\ = \mu_d \frac{N}{1+\partial_r u_t}, \end{aligned} \quad (20)$$

$$\eta_\phi = \mathcal{O}(1/N), \quad (21)$$

$$\eta_\sigma = \mu_d N \frac{\langle \partial_r \partial_s u_t \rangle^2}{(1 + \langle \partial_r u_t \rangle)^4} = \mathcal{O}(1). \quad (22)$$

In this model  $\eta_\sigma$  does *not* vanish in the large- $N$  limit, in contrast to quartic  $O(N)$  vector models where the anomalous dimensions of scalars vanish in the same limit [2].<sup>4</sup> Equation (20) implies that the effective potential  $u_\star$  at the RG fixed point, called a *scaling solution*, must satisfy

$$du_\star + (2-d)r\partial_r u_\star + \frac{1}{2}(2-d-\eta_\sigma)s\partial_s u_\star = \mu_d \frac{N}{1+\partial_r u_\star}. \quad (23)$$

Evidently there is a trivial solution  $u_\star = \mu_d N/d$  corresponding to the Gaussian fixed point for any  $N$  and  $d$ . Whether a globally well-defined nontrivial scaling solution to (23) [to be solved self-consistently with (22)] exists or not in  $d=5$  is our central concern here. The advantage of this FRG approach as compared to the  $\varepsilon$ -expansion is that one can search for a fixed point directly in  $d=5$  *without* placing any specific Ansatz for the effective potential. That said, it is usually hard to solve a fixed-point equation like (23) analytically. One may resort to solving it numerically, by integrating the partial differential equation starting from the origin. This

method reveals that most of numerical solutions thus obtained encounter a singularity at a finite value of the field, as emphasized by Morris [44, 48, 49] (see also [50, 51]). Even when the flow could be smoothly integrated over the entire field values, it may not be necessarily bounded from below. In these cases one has to conclude that there is no physical critical point. The main message here is that analyzing a truncated effective potential just around the origin is fallacious since it masks pathological global properties of the potential [16].

Now, coming back to (23), one finds that in the limit  $s \rightarrow 0$ ,

$$du_\star(r, 0) + (2-d)r\partial_r u_\star(r, 0) = \mu_d \frac{N}{1+\partial_r u_\star(r, 0)}, \quad (24)$$

which coincides exactly with the fixed-point equation with the optimized regulator for the quartic  $O(N)$  model with no  $\sigma$  field [41]. The structure of solutions to (24) for  $d > 4$  has already been thoroughly investigated in [16, 17] with the conclusion that they are either unbounded from below, or beset with singularities at a finite field.<sup>5</sup> The fact that  $u_\star(r, 0)$  is pathological forces us to conclude that (23) for  $d > 4$  possesses no acceptable scaling solution other than the trivial one. Note that the scalar  $\sigma$  plays no role here, although the cubic potential of  $\sigma$  in (2) appears at first sight to be the major source of instability in this model. The conclusion above may not come as a total surprise if we make the following observation: the scaling dimension of  $\sigma$  is equal to 2 at large  $N$  [6] so that both  $(\partial\sigma)^2$  and  $\sigma^3$  are irrelevant in  $d=5$  and can be dropped without affecting physics in IR. Then  $\sigma$  with the action  $\sim \phi_i \phi_i \sigma + \sigma^2$  can be integrated out, thus recovering the ordinary  $O(N)$  vector model with  $(\phi_i \phi_i)^2$  coupling.<sup>6</sup> Since the latter model does not possess a healthy scaling solution in  $d > 4$ , the cubic model (2) does not either.

### IV. DISCUSSION

In this paper, we have investigated a scalar  $O(N)$  model with cubic interactions using the functional renormalization group (FRG) method at next-to-leading order in the derivative expansion, for the purpose of testing a recent conjecture [6] (backed up by higher-spin AdS/CFT dualities [8, 9]) that there is an interacting unitary  $O(N)$ -symmetric CFT in  $d=5$  dimensions. The first analysis [6] based on the one-loop  $\varepsilon$ -expansion has already been extended to three [7] and even four loops [10], confirming that the cubic  $O(N)$  model has a non-Gaussian IR-stable

<sup>3</sup> IR fixed points at small  $N$  were also reported in [11] within  $O(N)$  models with tensorial interaction. Whether our fixed points  $A$  and  $B$  have anything to do with [11] is unclear.

<sup>4</sup> In fermionic theories, the nonvanishing  $\eta$  of scalars in the many-flavor limit is well known [46, 47].

<sup>5</sup> The lack of a lower bound for the potential is consistent with the fact that the UV fixed point value for the coupling  $(\phi_i \phi_i)^2$  in the quartic  $O(N)$  model with no  $\sigma$  is *negative* in  $d=4+\varepsilon$ , corresponding to a bottomless potential [2, 9].

<sup>6</sup> An analogous argument shows the equivalence between the Gross-Neveu model and the Yukawa model in  $2 < d < 4$  at large  $N$  [46, 52].

fixed point in  $d = 6 - \varepsilon$  dimensions if  $N$  is above a certain threshold ( $\lesssim 1000$ ). If true, this would herald new physics, defying the conventional wisdom that scalar theories in  $d \geq 4$  are trivial in the continuum limit. In order to place this claim on firmer ground one needs to rely on a nonperturbative approach. Preceding FRG analyses [16, 17] based on quartic  $O(N)$  vector models with  $N$  scalars have reported negative evidence as to the existence of a nontrivial stable fixed point, in harmony with earlier work [15]. On the other hand, in this work, we start directly from the cubic  $O(N)$  model considered in [6, 7]. We found that there is no IR-stable fixed point at large  $N$ , which corroborates [16, 17]. It is worth mentioning that we did not rely on a dimensional expansion from  $d = 4$  or  $6$  but directly worked in  $d = 5$ , and made no specific Ansatz for the effective potential to reach the above conclusion at large  $N$ . Thus we are led to conclude that the addition of a scalar  $\sigma$  with cubic interactions does not bring about qualitative differences from the quartic  $O(N)$  model. In this regard we disagree with the conformal bootstrap approach [12–14] which seems to be in favor of the putative fixed point.

Of course the analysis presented here is not completely free from approximations; we have used a truncated action at next-to-leading order in the derivative expansion. However, the anomalous dimension of  $\sigma$  ( $\sim 1/2$ ) is not so large as to invalidate the derivative expansion qualitatively. We wish to also mention that FRG at this level of approximation has been successful in many other circumstances [20, 22, 23, 25, 29]. If it transpires that the non-Gaussian fixed point does indeed exist, but is invisible in FRG, then it is an imperative task to understand why FRG fails to capture it. A deeper understanding of potential deficiencies of FRG would be instrumental in identifying the origin of discrepancy between FRG and other methods in fields such as QCD with two flavors [43, 45, 53–58] where the nature of the chiral transition is still under debate, and frustrated magnets [24, 56, 59–61] where the existence of IR fixed points is disputed. On the other hand, if the claimed  $O(N)$  critical theory is a non-unitary metastable theory (as is indicated by [16, 17] and this work), then a natural question to ask is how to distinguish such illusionary fixed points from physical ones within the conformal bootstrap approach. In either scenario our understanding of field theories can be deepened through a further investigation on this issue.

Last but not least, the (non)existence of stable fixed points at finite or small  $N$  in  $d > 4$  is also of interest.

In this work, we have found two IR-stable non-Gaussian fixed points in the real-coupling region for  $N = 2$  and  $d = 5$ . This should not be taken at face value, however, given the large anomalous dimensions at these points which are likely to be a signal of the breakdown of the derivative expansion. While we have not attempted to explore the domain of imaginary couplings, the latter has physical importance with regards to e.g., the Yang-Lee edge singularity [30], percolation problems [32] and  $\mathcal{PT}$ -symmetry [35]. For these applications our flow equation (5) provides a useful point of departure for a nonperturbative analysis in the future.

### Note added

While this paper was at the final stage of preparation, we became aware of independent work [62] where the same model was analyzed.

## ACKNOWLEDGMENTS

T. K. was supported by the RIKEN iTHES project.

## Appendix: Flow equation

In this appendix we derive the flow equations (5), (7) and (9) for the effective potential  $U_k$  and the wave function renormalization  $Y_k$  and  $Z_k$ .

### 1. Flow of $U_k$

In a homogeneous background, (1) may be evaluated in a plane wave basis:

$$\partial_k U_k = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \text{tr} \left[ \frac{1}{\Gamma_k^{(2)} + R_k} \partial_k R_k \right], \quad (\text{A.1})$$

with  $R_k = \text{diag}(R_k^\sigma, R_k^\phi \mathbb{1}_N)$ , see (4). Note that both  $\Gamma_k^{(2)}$  and  $R_k$  are  $(N+1) \times (N+1)$  matrices in the space of field components. Without loss of generality, any  $\vec{\phi}$  can be rotated to the first direction as  $\vec{\phi} = (\phi, 0, \dots, 0)$  so that  $\rho \equiv \vec{\phi}^2/2 = \phi^2/2$ . Then

$$\Gamma_k^{(2)} + R_k = \begin{pmatrix} Z_k p^2 + R_k^\sigma + \frac{\partial^2 U_k}{\partial \sigma^2} & \phi \frac{\partial^2 U_k}{\partial \rho \partial \sigma} & 0 \\ \phi \frac{\partial^2 U_k}{\partial \rho \partial \sigma} & Y_k p^2 + R_k^\phi + \frac{\partial U_k}{\partial \rho} + \phi^2 \frac{\partial^2 U_k}{\partial \rho^2} & 0 \\ 0 & 0 & (Y_k p^2 + R_k^\phi + \frac{\partial U_k}{\partial \rho}) \mathbb{1}_{N-1} \end{pmatrix}. \quad (\text{A.2})$$



Plugging this into (A.1), one finds

$$\partial_k U_k = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \left[ X_\sigma \partial_k R_k^\sigma + X_\phi \partial_k R_k^\phi + (N-1) \frac{\partial_k R_k^\phi}{Y_k p^2 + R_k^\phi + \frac{\partial U_k}{\partial \rho}} \right] \quad (\text{A.3})$$

with

$$X_\sigma \equiv \frac{Y_k p^2 + R_k^\phi + \frac{\partial U_k}{\partial \rho} + \phi^2 \frac{\partial^2 U_k}{\partial \rho^2}}{(Z_k p^2 + R_k^\sigma + \frac{\partial^2 U_k}{\partial \sigma^2})(Y_k p^2 + R_k^\phi + \frac{\partial U_k}{\partial \rho} + \phi^2 \frac{\partial^2 U_k}{\partial \rho^2}) - \phi^2 (\frac{\partial^2 U_k}{\partial \rho \partial \sigma})^2}, \quad (\text{A.4a})$$

$$X_\phi \equiv \frac{Z_k p^2 + R_k^\sigma + \frac{\partial^2 U_k}{\partial \sigma^2}}{(Z_k p^2 + R_k^\sigma + \frac{\partial^2 U_k}{\partial \sigma^2})(Y_k p^2 + R_k^\phi + \frac{\partial U_k}{\partial \rho} + \phi^2 \frac{\partial^2 U_k}{\partial \rho^2}) - \phi^2 (\frac{\partial^2 U_k}{\partial \rho \partial \sigma})^2}. \quad (\text{A.4b})$$

This rather complicated appearance is caused by the mixing between  $\rho$  and  $\sigma$ . A simplification comes from the observation that the presence of  $\partial_k R_k^\sigma$  and  $\partial_k R_k^\phi$  in (A.3) allows us to replace  $R_k^\sigma$  and  $R_k^\phi$  in  $X_{\sigma,\phi}$  by  $Z_k(k^2 - p^2)$  and  $Y_k(k^2 - p^2)$ , respectively. This leads to the expression

$$\begin{aligned} \partial_k U_k = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} & \left[ \frac{(Y_k k^2 + \frac{\partial U_k}{\partial \rho} + 2\rho \frac{\partial^2 U_k}{\partial \rho^2}) \partial_k R_k^\sigma}{(Z_k k^2 + \frac{\partial^2 U_k}{\partial \sigma^2})(Y_k k^2 + \frac{\partial U_k}{\partial \rho} + 2\rho \frac{\partial^2 U_k}{\partial \rho^2}) - 2\rho (\frac{\partial^2 U_k}{\partial \rho \partial \sigma})^2} \right. \\ & \left. + \frac{(Z_k k^2 + \frac{\partial^2 U_k}{\partial \sigma^2}) \partial_k R_k^\phi}{(Z_k k^2 + \frac{\partial^2 U_k}{\partial \sigma^2})(Y_k k^2 + \frac{\partial U_k}{\partial \rho} + 2\rho \frac{\partial^2 U_k}{\partial \rho^2}) - 2\rho (\frac{\partial^2 U_k}{\partial \rho \partial \sigma})^2} + (N-1) \frac{\partial_k R_k^\phi}{Y_k k^2 + \frac{\partial U_k}{\partial \rho}} \right]. \end{aligned} \quad (\text{A.5})$$

In the above we replaced  $\phi^2$  by  $2\rho$ . Now the remaining integral over  $p$  can be easily done with the formulas

$$\int \frac{d^d p}{(2\pi)^d} \partial_k R_k^\sigma = 2\mu_d k^{d+1} \left( \frac{1}{d+2} k \partial_k Z_k + Z_k \right), \quad (\text{A.6a})$$

$$\int \frac{d^d p}{(2\pi)^d} \partial_k R_k^\phi = 2\mu_d k^{d+1} \left( \frac{1}{d+2} k \partial_k Y_k + Y_k \right), \quad (\text{A.6b})$$

with  $\mu_d$  defined in (10), which finally yields the flow equation (5) for  $U_k$ .

## 2. Flow of $Y_k$ and $Z_k$

Let us evaluate (1) in an inhomogeneous background

$$(\sigma | \phi_1, \dots, \phi_N) = (\sigma_0 + u(x) | t(x), 0, \dots, 0) \quad (\text{A.7})$$

for which  $\rho \equiv \vec{\phi}^2/2 = t^2/2$ . Then the matrix elements of  $\Gamma_k^{(2)}$  admit an expansion in powers of  $u$  and  $t$  around  $\sigma_0$ . For instance

$$\begin{aligned} \frac{\partial^2 U_k}{\partial \sigma \partial \rho} &= \left\langle \frac{\partial^2 U_k}{\partial \sigma \partial \rho} \right\rangle + \left\langle \frac{\partial^3 U_k}{\partial \sigma^2 \partial \rho} \right\rangle u + \frac{1}{2} \left\langle \frac{\partial^4 U_k}{\partial \sigma^3 \partial \rho} \right\rangle u^2 \\ &+ \frac{1}{2} \left\langle \frac{\partial^3 U_k}{\partial \sigma \partial \rho^2} \right\rangle t^2 + \mathcal{O}(u^3, ut^2), \end{aligned} \quad (\text{A.8})$$

where  $\langle \dots \rangle$  is to be evaluated at  $(\sigma, \vec{\phi}) = (\sigma_0, \vec{0})$  and we exploited the fact that terms odd in  $t$  do not show up in the expansion. This way we obtain

$$\Gamma_k^{(2)} + R_k = A_0 + A_1 + A_2 + \mathcal{O}(u^3, u^2 t, ut^2, t^3), \quad (\text{A.9})$$

where  $A_{0,1,2}$  are  $(N+1) \times (N+1)$  matrices in the field space, defined as

$$A_0 \equiv \begin{pmatrix} -Z_k \partial^2 + R_k^\sigma + \left\langle \frac{\partial^2 U_k}{\partial \sigma^2} \right\rangle & 0 \\ 0 & [-Y_k \partial^2 + R_k^\phi + \left\langle \frac{\partial U_k}{\partial \rho} \right\rangle] \mathbb{1}_N \end{pmatrix}, \quad (\text{A.10})$$

$$A_1 \equiv \begin{pmatrix} u \left\langle \frac{\partial^3 U_k}{\partial \sigma^3} \right\rangle & t \left\langle \frac{\partial^2 U_k}{\partial \sigma \partial \rho} \right\rangle & 0 \\ t \left\langle \frac{\partial^2 U_k}{\partial \sigma \partial \rho} \right\rangle & u \left\langle \frac{\partial^2 U_k}{\partial \sigma \partial \rho} \right\rangle & 0 \\ 0 & 0 & u \left\langle \frac{\partial^2 U_k}{\partial \sigma \partial \rho} \right\rangle \mathbb{1}_{N-1} \end{pmatrix}, \quad (\text{A.11})$$

and  $A_2$  is a collection of terms at  $\mathcal{O}(u^2, ut, t^2)$ . Defining  $\tilde{\partial}_k$  as a derivative acting only on the  $k$ -dependence of  $R_k$ , we obtain

$$\begin{aligned} \partial_k \Gamma_k \Big|_{\text{kin}} &= \frac{1}{2} \tilde{\partial}_k \text{Tr} \log(\Gamma_k^{(2)} + R_k) \Big|_{\text{kin}} \\ &= \frac{1}{2} \tilde{\partial}_k \text{Tr} \log(A_0 + A_1 + A_2 + \dots) \Big|_{\text{kin}} \\ &= \frac{1}{2} \tilde{\partial}_k \text{Tr} \left[ \log A_0 + \log \left\{ \mathbb{1} + A_0^{-1} (A_1 + A_2 + \dots) \right\} \right] \Big|_{\text{kin}} \\ &= \frac{1}{2} \tilde{\partial}_k \text{Tr} \left[ -\frac{1}{2} A_0^{-1} A_1 A_0^{-1} A_1 \right] \Big|_{\text{kin}}, \end{aligned} \quad (\text{A.12})$$

where we have used that  $\text{Tr}[\log A_0]$  and  $\text{Tr}[A_0^{-1} A_2]$  do not contribute to the kinetic term.

On the other hand, a direct substitution of (A.7) into the Ansatz (3) yields

$$\partial_k \Gamma_k \Big|_{\text{kin}} = \frac{1}{2} \int_q [(\partial_k Y_k) q^2 t_{q^2-q} + (\partial_k Z_k) q^2 u_q u_{-q}]. \quad (\text{A.13})$$

Juxtaposing (A.12) with (A.13), we obtain

$$\partial_k Y_k = - \left\langle \frac{\partial^2 U_k}{\partial \sigma \partial \rho} \right\rangle^2 \lim_{q \rightarrow 0} \frac{\partial}{\partial(q^2)} \tilde{\partial}_k \int_p \frac{1}{Z_k p^2 + R_k^\sigma(p) + \left\langle \frac{\partial^2 U_k}{\partial \sigma^2} \right\rangle} \frac{1}{Y_k(p+q)^2 + R_k^\phi(p+q) + \left\langle \frac{\partial U_k}{\partial \rho} \right\rangle}, \quad (\text{A.14})$$

$$\begin{aligned} \partial_k Z_k = & -\frac{1}{2} \left\langle \frac{\partial^3 U_k}{\partial \sigma^3} \right\rangle^2 \lim_{q \rightarrow 0} \frac{\partial}{\partial(q^2)} \tilde{\partial}_k \int_p \frac{1}{Z_k(q+p)^2 + R_k^\sigma(q+p) + \left\langle \frac{\partial^2 U_k}{\partial \sigma^2} \right\rangle} \frac{1}{Z_k p^2 + R_k^\sigma(p) + \left\langle \frac{\partial^2 U_k}{\partial \sigma^2} \right\rangle} \\ & - \frac{N}{2} \left\langle \frac{\partial^2 U_k}{\partial \sigma \partial \rho} \right\rangle^2 \lim_{q \rightarrow 0} \frac{\partial}{\partial(q^2)} \tilde{\partial}_k \int_p \frac{1}{Y_k(q+p)^2 + R_k^\phi(q+p) + \left\langle \frac{\partial U_k}{\partial \rho} \right\rangle} \frac{1}{Y_k p^2 + R_k^\phi(p) + \left\langle \frac{\partial U_k}{\partial \rho} \right\rangle}. \end{aligned} \quad (\text{A.15})$$

Finally we evaluate (A.14) and (A.15) analytically with the help of formulas for threshold functions with the op-

timized regulator in e.g., [63, 64]. This leads to the relatively simple expressions, (7) and (9).

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- [1] N. Goldenfeld, *Lectures on phase transitions and the renormalization group*, (Frontiers in physics, 85) (Addison-Wesley, Reading, 1992).
  - [2] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, 4th ed. (Clarendon Press, Oxford, 2002).
  - [3] A. Pelissetto and E. Vicari, *Phys. Rept.* **368**, 549 (2002), [arXiv:cond-mat/0012164 \[cond-mat\]](#).
  - [4] J. Frohlich, *Nucl. Phys.* **B200**, 281 (1982).
  - [5] M. Aizenman, *Commun. Math. Phys.* **86**, 1 (1982).
  - [6] L. Fei, S. Giombi, and I. R. Klebanov, *Phys. Rev.* **D90**, 025018 (2014), [arXiv:1404.1094 \[hep-th\]](#).
  - [7] L. Fei, S. Giombi, I. R. Klebanov, and G. Tarnopolsky, *Phys. Rev.* **D91**, 045011 (2015), [arXiv:1411.1099 \[hep-th\]](#).
  - [8] I. R. Klebanov and A. M. Polyakov, *Phys. Lett.* **B550**, 213 (2002), [arXiv:hep-th/0210114 \[hep-th\]](#).
  - [9] S. Giombi, I. R. Klebanov, and B. R. Safdi, *Phys. Rev.* **D89**, 084004 (2014), [arXiv:1401.0825 \[hep-th\]](#).
  - [10] J. A. Gracey, *Phys. Rev.* **D92**, 025012 (2015), [arXiv:1506.03357 \[hep-th\]](#).
  - [11] I. F. Herbut and L. Janssen, *Phys. Rev.* **D93**, 085005 (2016), [arXiv:1510.05691 \[hep-th\]](#).
  - [12] Y. Nakayama and T. Ohtsuki, *Phys. Lett.* **B734**, 193 (2014), [arXiv:1404.5201 \[hep-th\]](#).
  - [13] J.-B. Bae and S.-J. Rey, (2014), [arXiv:1412.6549 \[hep-th\]](#).
  - [14] S. M. Chester, S. S. Pufu, and R. Yacoby, *Phys. Rev.* **D91**, 086014 (2015), [arXiv:1412.7746 \[hep-th\]](#).
  - [15] O. J. Rosten, *JHEP* **07**, 019 (2009), [arXiv:0808.0082 \[hep-th\]](#).
  - [16] R. Percacci and G. P. Vacca, *Phys. Rev.* **D90**, 107702 (2014), [arXiv:1405.6622 \[hep-th\]](#).
  - [17] P. Mati, *Phys. Rev.* **D91**, 125038 (2015), [arXiv:1501.00211 \[hep-th\]](#).
  - [18] J. F. Nicoll and T. S. Chang, *Phys. Lett.* **A62**, 287 (1977).
  - [19] C. Wetterich, *Phys. Lett.* **B301**, 90 (1993).
  - [20] T. R. Morris, *Int. J. Mod. Phys.* **A9**, 2411 (1994), [arXiv:hep-ph/9308265 \[hep-ph\]](#).
  - [21] T. R. Morris, *Phys. Lett.* **B329**, 241 (1994), [arXiv:hep-ph/9403340 \[hep-ph\]](#).
  - [22] C. Bagnuls and C. Bervillier, *Phys. Rept.* **348**, 91 (2001), [arXiv:hep-th/0002034 \[hep-th\]](#).
  - [23] J. Berges, N. Tetradis, and C. Wetterich, *Phys. Rept.* **363**, 223 (2002), [arXiv:hep-ph/0005122 \[hep-ph\]](#).
  - [24] B. Delamotte, D. Mouhanna, and M. Tissier, *Phys. Rev.* **B69**, 134413 (2004), [arXiv:cond-mat/0309101 \[cond-mat\]](#).
  - [25] B. Delamotte, *Lect. Notes Phys.* **852**, 49 (2012), [arXiv:cond-mat/0702365 \[cond-mat.stat-mech\]](#).
  - [26] O. J. Rosten, *Phys. Rept.* **511**, 177 (2012), [arXiv:1003.1366 \[hep-th\]](#).
  - [27] J. Braun, *J. Phys.* **G39**, 033001 (2012), [arXiv:1108.4449 \[hep-ph\]](#).
  - [28] A. Wipf, *Lect. Notes Phys.* **864** (2013), 10.1007/978-3-642-33105-3.
  - [29] S. Nagy, *Annals Phys.* **350**, 310 (2014), [arXiv:1211.4151 \[hep-th\]](#).
  - [30] M. E. Fisher, *Phys. Rev. Lett.* **40**, 1610 (1978).
  - [31] O. F. de Alcantara Bonfim, J. E. Kirkham, and A. J. McKane, *J. Phys.* **A14**, 2391 (1981).
  - [32] C. M. Fortuin and P. W. Kasteleyn, *Physica* **57**, 536 (1972).
  - [33] B. Grinstein, D. Stone, A. Stergiou, and M. Zhong, *Phys.*

- Rev. Lett. **113**, 231602 (2014), arXiv:1406.3626 [hep-th].
- [34] B. Grinstein, A. Stergiou, D. Stone, and M. Zhong, Phys. Rev. **D92**, 045013 (2015), arXiv:1504.05959 [hep-th].
  - [35] C. M. Bender, V. Branchina, and E. Messina, Phys. Rev. **D85**, 085001 (2012), arXiv:1201.1244 [hep-th].
  - [36] C. M. Bender, V. Branchina, and E. Messina, Phys. Rev. **D87**, 085029 (2013), arXiv:1301.6207 [hep-th].
  - [37] N. Tetradis and C. Wetterich, Nucl.Phys. **B422**, 541 (1994), arXiv:hep-ph/9308214 [hep-ph].
  - [38] M. D’Attanasio and T. R. Morris, Phys. Lett. **B409**, 363 (1997), arXiv:hep-th/9704094 [hep-th].
  - [39] M. Moshe and J. Zinn-Justin, Phys. Rept. **385**, 69 (2003), arXiv:hep-th/0306133 [hep-th].
  - [40] D. F. Litim, Phys. Lett. **B486**, 92 (2000), arXiv:hep-th/0005245 [hep-th].
  - [41] D. F. Litim, Phys.Rev. **D64**, 105007 (2001), arXiv:hep-th/0103195 [hep-th].
  - [42] K.-I. Aoki, K. Morikawa, W. Souma, J.-I. Sumi, and H. Terao, Prog. Theor. Phys. **99**, 451 (1998), arXiv:hep-th/9803056 [hep-th].
  - [43] K. Fukushima, K. Kamikado, and B. Klein, Phys. Rev. **D83**, 116005 (2011), arXiv:1010.6226 [hep-ph].
  - [44] T. R. Morris, Phys. Lett. **B334**, 355 (1994), arXiv:hep-th/9405190 [hep-th].
  - [45] M. Grah, Phys. Rev. **D90**, 117904 (2014), arXiv:1410.0985 [hep-th].
  - [46] J. Zinn-Justin, Nucl. Phys. **B367**, 105 (1991).
  - [47] J. Braun, H. Gies, and D. D. Scherer, Phys. Rev. **D83**, 085012 (2011), arXiv:1011.1456 [hep-th].
  - [48] T. R. Morris, Phys. Rev. Lett. **77**, 1658 (1996), arXiv:hep-th/9601128 [hep-th].
  - [49] T. R. Morris, Prog. Theor. Phys. Suppl. **131**, 395 (1998), arXiv:hep-th/9802039 [hep-th].
  - [50] A. Codello, J. Phys. **A45**, 465006 (2012), arXiv:1204.3877 [hep-th].
  - [51] T. Hellwig, A. Wipf, and O. Zanusso, Phys. Rev. **D92**, 085027 (2015), arXiv:1508.02547 [hep-th].
  - [52] A. Hasenfratz, P. Hasenfratz, K. Jansen, J. Kuti, and Y. Shen, Nucl. Phys. **B365**, 79 (1991).
  - [53] R. D. Pisarski and F. Wilczek, Phys.Rev. **D29**, 338 (1984).
  - [54] S. Aoki, H. Fukaya, and Y. Taniguchi, Phys.Rev. **D86**, 114512 (2012), arXiv:1209.2061 [hep-lat].
  - [55] A. Pelissetto and E. Vicari, Phys. Rev. **D88**, 105018 (2013), arXiv:1309.5446 [hep-lat].
  - [56] Y. Nakayama and T. Ohtsuki, Phys. Rev. **D91**, 021901 (2015), arXiv:1407.6195 [hep-th].
  - [57] T. Kanazawa and N. Yamamoto, JHEP **01**, 141 (2016), arXiv:1508.02416 [hep-th].
  - [58] B. Delamotte, M. Dudka, D. Mouhanna, and S. Yabunaka, Phys. Rev. B **93**, 064405 (2016), arXiv:1510.00169 [cond-mat.stat-mech].
  - [59] H. Kawamura, Journal of Physics: Condensed Matter **10**, 4707 (1998), cond-mat/9805134.
  - [60] A. Butti, A. Pelissetto, and E. Vicari, JHEP **08**, 029 (2003), arXiv:hep-ph/0307036 [hep-ph].
  - [61] P. Calabrese, P. Parruccini, A. Pelissetto, and E. Vicari, Phys. Rev. **B70**, 174439 (2004), arXiv:cond-mat/0405667 [cond-mat].
  - [62] A. Eichhorn, L. Janssen, and M. M. Scherer, (2016), arXiv:1604.03561 [hep-th].
  - [63] F. Hofling, C. Nowak, and C. Wetterich, Phys.Rev. **B66**, 205111 (2002), arXiv:cond-mat/0203588 [cond-mat].
  - [64] J. Braun, Eur.Phys.J. **C64**, 459 (2009), arXiv:0810.1727 [hep-ph].